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# TECHNICAL NOTES

## Temperature distribution in a cylindrical conductor with skin effect

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### INTRODUCTION

THE PROBLEM of heat transfer in electrical conductors with skin effect was examined a long time ago by Strutt [1]. This author considered particular applications occurring in the treatment of electric cables, lines and conductive heating. His method was based on solving analytically, a linear problem but by considering asymptotic limits of small and large skin effect only. The same problem has been re-examined more recently by Mytkowski and Rozanski [2] who solved the linear problem for an arbitrary skin effect using a numerical method. We may also mention the work of Rolicz [3] who solved the linear heat transfer equation for the case of an electrical alternating current in a cylindrical conductor but without taking into account explicitly the skin effect. In a series of recent papers [4–6] we have investigated the problem of non-linear heat transfer in cylindrical conductors with a constant current density, and hence without introducing the skin effect phenomenon. Our purpose was to investigate the influence of the temperature dependence of various physical parameters such as specific heat, thermal conductivity, electrical resistivity, etc. on the temperature distribution in cylindrical conductors both in space and time. Here we would like to extend this work to cases where the current density is a complex quantity and, therefore, introducing the skin effect. We will examine the linear problem only, where the physical parameters mentioned above do not depend on temperature. The reason is that, in order to solve the non-linear problem of the temperature distribution one needs the solution of the corresponding linear case. Our method is based on an analytical solution of the temperature distribution equation with arbitrary values of the skin effect.

In the linear steady-state problem, we need to solve the following simple equation:

$$\lambda\left[\frac{\mathrm{d}^2T}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}T}{\mathrm{d}r}\right]+\rho J^2=0 \tag{1}$$

where $\lambda$, $\rho$ and $J$ are the thermal conductivity, the electrical resistivity and the current density in the $z$-direction, respectively. Here $T$ represents the increase of temperature above the ambient temperature, and $r$ the distance from the centre line of the conductor. In our previous studies, we always assumed $J$ to be constant, independent of $r$. Here we note the maximum current density by $J_\mathrm{m}$ and we assume that it is a complex quantity and a function of $r$. In this case equation (1) becomes [3]

$$\frac{\mathrm{d}^2T}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}T}{\mathrm{d}r}+\frac{\rho}{2\lambda}|J_\mathrm{m}|^2=0 \tag{2}$$

where $T$ represents the mean temperature as defined by Rolicz [3].

This leads us to introduce the so-called skin effect phenomenon. Using Maxwell's equations, one can obtain the following differential equation for $J_\mathrm{m}$:

$$\frac{\mathrm{d}^2J_\mathrm{m}}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}J_\mathrm{m}}{\mathrm{d}r}+\xi J_\mathrm{m}=0 \tag{3}$$

where $\xi$ is a complex parameter that depends on the physical properties of the system, as well as on the current's frequency $f$. Hence, one must bear in mind that $J_\mathrm{m}$ may be a complex quantity. The problem now is to solve both equations (2) and (3) using specific boundary conditions. These are chosen as follows:

$$\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=0}=0 \tag{4}$$

$$\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{r=r_0}=-\frac{\varepsilon}{\lambda}T(r_0) \tag{5}$$

where $r_0$ is the radius of the conductor and $\varepsilon$ the convective heat transfer coefficient. For equation (3), governing the variation with $r$ of the complex current density, one uses Ampere's law to obtain the corresponding boundary condition. The problem presented here was examined in ref. [2] where the following equation was solved:

$$\frac{\mathrm{d}^2T}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}T}{\mathrm{d}r}=-\frac{I_\mathrm{m}^2p}{8\pi^2r_0^2\gamma\lambda}\frac{\mathrm{ber}^2\,(\sqrt{p}r)+\mathrm{bei}^2\,(\sqrt{p}r)}{\mathrm{ber}_1^2\,(\sqrt{p}r_0)+\mathrm{bei}_1^2\,(\sqrt{p}r_0)} \tag{6}$$

where $I_\mathrm{m}$ is the maximum current, $\gamma=\rho^{-1}$ is the electrical conductivity, $p=2\pi f\mu\gamma$, and $\mu$ is the magnetic permeability. The special functions involved in this equation are defined as [7]

$$\mathrm{ber}_\nu\,x=\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}\quad\cos 3/4(\nu+2n)\pi \tag{7}$$

$$\mathrm{bei}_\nu\,x=\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}\sin 3/4(\nu+2n)\pi, \tag{8}$$

$$(\nu=0,1).$$

Equation (6) was solved numerically in ref. [2].

In the present work we would like to examine the same problem using a different method which leads to a relatively simple analytical expression for $T(r)$. The reason for seeking such a solution was explained earlier. The next section is devoted to the description of the general formalism of our method and to specific applications.

### GENERAL FORMALISM AND APPLICATIONS

The general problem of interest in this work is the resolution of the set of equations (2) and (3). For this purpose, let us examine first the following equation for an arbitrary complex function $F(r)$.

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}+\xi F=0. \tag{9}$$

Taking the complex conjugate of each term, denoted by an asterisk, one produces the following equation:

$$\frac{\mathrm{d}^2F^*}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}F^*}{\mathrm{d}r}+\xi^*F^*=0. \tag{10}$$

We define

$$K(r) = F(r) \cdot F^*(r) \tag{11}$$

and

$$H(r) = \frac{dF(r)}{dr} \cdot \frac{dF^*(r)}{dr}. \tag{12}$$

Multiplying equation (9) by $F^*$ and equation (10) by $F$ and adding them, one obtains after some rearrangements, the following equations:

$$\frac{d^2K}{dr^2} + \frac{1}{r}\frac{dK}{dr} + 2\alpha K = 2H \tag{13}$$

$$\frac{d^2H}{dr^2} + \frac{3}{r}\frac{dH}{dr} + 2\alpha H = 2|\xi|^2 K \tag{14}$$

where $\alpha$ is the real part of $\xi$.

To proceed further, we assume that the solution can be written as

$$K(r) = \sum_{n=0}^{\infty} a_{2n} r^{2n} \tag{15}$$

$$H(r) = \sum_{n=0}^{\infty} b_{2n} r^{2n} \tag{16}$$

where we have retained the even powers of $r$ only for reasons of symmetry. Substituting these expressions into equations (13) and (14), collecting the terms having the same powers of $r$, letting $\alpha = 0$, and setting $a_0 = 1$, $b_0 = 0$ and $b_2 = |\xi|^2/4$ leads to the following recurrence relations for the constant coefficient defined in equations (15) and (16)

$$(2n+4)^2 a_{2n+4} = 2b_{n+2} \tag{17a}$$

$$(2n+2)(2n+4)b_{2n+2} = 2|\xi|^2 a_{2n}. \tag{17b}$$

Using these relations in equation (15) it gives

$$K(r) = 1 + \sum_{n=1}^{\infty} \frac{1}{(n!)^3(2n-1)!!} \times \left(\frac{|\xi|^{1/2} r}{2^{5/4}}\right)^{4n}. \tag{18}$$

Here we do not write $H(r)$ because it is not needed and $n!! = 1 \cdot 3 \cdot 5 \ldots$ Let us now return to our problem of solving equation (2) for the temperature distribution. It can be written explicitly as

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \Gamma K(r) = 0 \tag{19}$$

where $\Gamma$ is given by

$$\Gamma = \frac{\rho}{2\lambda} \frac{I_m^2 P}{4\pi^2 r_0^2} \frac{1}{(\mathrm{ber}'\sqrt{p}r_0)^2 + (\mathrm{bei}'\sqrt{p}r_0)^2} \tag{20}$$

and $I_m$ is the magnitude of the current, $p = |\xi|$. This is a classical second-order linear differential equation which can be solved by a standard method. We observe that the general solution of the homogeneous equation can be written as

$$C_1 + C_2 \ln r$$

where $C_1$ and $C_2$ are constants. Moreover, the particular solution of the non-homogeneous equation can be assumed to be of the form

$$U(r) = \sum_{n=1}^{\infty} u_{2n} r^{2n}. \tag{21}$$

Using the boundary condition in equation (5), one obtains the constant $C_1$ as follows:

$$C_1 = -\left[U(r_0) + \frac{\lambda}{\varepsilon} U'(r_0)\right]. \tag{22}$$

One also obtains $C_2 = 0$ because the temperature must remain finite at $r = 0$ (see equation (4)).

Letting the general solution for $T(r)$ be

$$T(r) = U(r) - \left[U(r_0) + \frac{\lambda}{\varepsilon} U'(r_0)\right] \tag{23}$$

one obtains

$$T(r) = \Gamma\left[r_0^2 A(r_0/R) + \frac{\lambda r_0}{\varepsilon} B(r_0/R) - r^2 A(r/R)\right] \tag{24}$$

where $R = 2^{5/4}/p^{1/2}$ and the functions $A(r/R)$, $B(r/R)$ are defined by

$$A(r/R) = 0.25 + \sum_{n=1}^{\infty} \frac{1}{(n!)^3(2n-1)!!(4n+2)^2}(r/R)^{4n}$$
$$B(r/R) = 0.5 + \sum_{n=1}^{\infty} \frac{1}{(n!)^3(2n-1)!!(4n+2)}(r/R)^{4n}. \tag{25}$$

*Applications*

To illustrate these results we have considered the case of a cylindrical conductor made of tungsten and characterized by the following parameters [8]:

$r_0 = 3 \times 10^{-3}$ m

$\rho = 92 \times 10^{-8}\,\Omega$m at 3000 K

$\lambda = 95$ W m$^{-1}$ K$^{-1}$ at 3000 K

$f = 900$ kHz and 50 kHz

$J_m = 1115.5\sqrt{2}$ A and $1930.8\sqrt{2}$ A.

Figure 1 displays the variation of the temperature as a function of $r$ for two different cases $r_0\sqrt{p} = 8.3376$ and 1.965 corresponding to a strong (curve 1) and weak (curve 2) skin effect, respectively. One observes that in the case of the strong skin effect, the temperature within the conductor drops rapidly near the surface.

As a test of our result in equation (23), we have solved numerically equation (2) using the Runge–Kutta method [4, 5]. The comparison between the analytical and numerical solutions shows a good agreement as one can observe from Fig. 1.

## CONCLUSION

The present work is devoted to the study of linear heat transfer problems in cylindrical conductors with arbitrary

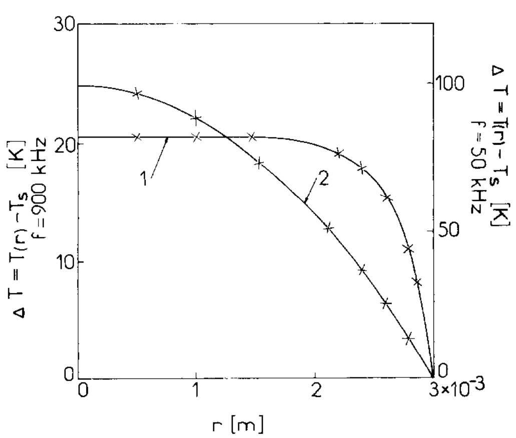


FIG. 1. The variation of the temperature increase $T$ as a function of $r$ in two cases: curve 1, $f = 900$ kHz strong skin effect; curve 2, $f = 50$ kHz weak skin effect. Here the surface temperatures of the conductor are chosen as $T_{s,1} = 3003.06$ K for curve 1 and $T_{s,2} = 3003.22$ K for curve 2.

skin effect. Its purpose is essentially to develop an analytical method for solving such problems, because the analytical solution is useful in the treatment of non-linear problems as we shall see later.

Dynamical variations of the temperature can also be examined within the framework of this theory. These generalizations are currently under investigation and the results will be reported elsewhere.

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# Forced convection heat transfer in smooth tubes roughened by helically coiled ribbons

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## INTRODUCTION

With increasing emphasis on economic energy saving considerations, efforts are being made to develop better heat transfer surfaces to produce more efficient heat exchange equipment. Internal roughness such as sand-grain textures [1, 2], internal ribbing [3, 4] and spirally corrugated tube surfaces [5] have been studied or applied with varying degrees of success.

Helically coiled wires [6] or ribbons, fitted tightly inside smooth tubes, give a considerable increase in heat transfer rate without a significant increase in friction power, as these tubes produce some helical flow at the periphery of flow, superimposed upon the main axial flow, and thus influence the velocity distribution, the turbulence level and the turbulent wall shear. As no previous study has been made on smooth tubes, roughened with coiled ribbons, the present investigation was carried out to study their frictional and heat transfer performance and develop suitable correlations for momentum and heat transfer roughness functions, based on friction and heat transfer similarity laws.

## THEORETICAL BACKGROUND

Nikuradse [1] used the law of the wall concept and obtained the turbulent flow velocity distributions for the smooth and rough tubes

$$u^+ = 2.5\ln y^+ + 5.5 \quad \text{(smooth tube)} \tag{1}$$

$$u^+ = 2.5\ln(y/h) + R(h^+) \quad \text{(rough tube).} \tag{2}$$

On integration over the tube cross-section, equation (2) gives

$$R(h^+) = \sqrt{(2/f)} + 2.5\ln(2h/D) + 3.75. \tag{3}$$

For the fully rough region ($h^+ > 70$), $R(h^+)$ attained a constant value of 8.48 for the sand-grain rough tubes of Nikuradse [1] and Dipprey and Sabersky [2]. Results of turbulent friction factors expressed as $R(h^+)$ were successfully correlated by Webb *et al.* [3] for tubes with transverse ribs, and by Ganeshan and Raja Rao [5] for spirally corrugated tubes.

Dipprey and Sabersky [2] first developed a heat transfer similarity law, analogous to the friction similarity law and correlated their heat transfer results in terms of Prandtl number and roughness Reynolds number

$$G(h^+, Pr) = \left[\left(\frac{f}{2St} - 1\right)\Big/\sqrt{(f/2)} + R(h^+)\right] = 5.19(Pr)^{0.44}(h^+)^{0.20}. \tag{4}$$

The recent work of Ganeshan and Raja Rao [5] and Gee and Webb [4] indicates that the heat transfer similarity law can be applied to other rough surfaces, having discrete two-dimensional roughness elements.

## EXPERIMENTAL WORK

The three helically coiled ribbons were fabricated by winding a long strip of copper sheet (0.72 mm thick and 4.5 mm wide) on a cylindrical rod of 23.5 mm diameter, using a precision lathe. Thin line impressions of the desired helical paths were engraved on the rods, before the winding operations. The pitch of the coiled ribbons used was 41, 21 and 11 mm, corresponding to helix angles of 51°, 66° and 79°, respectively. A cut section of the smooth tube, roughened by a typical coiled ribbon is shown in Fig. 1, and geometrical properties of the tubes are listed in Table 1. The ribbon coils, when introduced into the smooth tube, fitted tightly, ensuring close contact between the ribbon surface and tube wall.

The apparatus for this work is the same as used in our earlier study [6], and consisted of a 2050 mm long double-pipe heat exchanger, along with auxiliary equipment for circulation of hot (test) liquid on the tube side, and cold water on the annulus side, in closed loops. Water and 40% aqueous glycerol were used as the test liquids.